Combinatorial Hopf algebras in particle physics III

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1 Renormalization Group (continued)

Lemma 1. *If $\phi : \mathcal{H} \to \mathbb{Q}[x]$ is a morphism of Hopf algebras ($\phi(ab) = \phi(a)\phi(b)$, $\Delta\phi = (\phi \otimes \phi)\Delta$) then $\phi = exp_*(xd_0 \circ \phi)$, $\phi(f) = \sum a_n x^n$, $d_0(\phi(f)) = a_1$.*

We call $\gamma := -d_0\phi_R$ the *anomalous dimension.*

The following example summarizes what we did last lecture.

Example. Recall $\gamma(\mathbf{1}) = 0$, $\gamma(ab) = 0$ unless $a = \mathbf{1}$ or $b = \mathbf{1}$; $\gamma(f) \neq 0$ if and only if f is a tree, not a forest.

$$\begin{aligned}
\phi_R\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) &= \exp_*(-l\gamma)\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) = \left(e - l\gamma + \frac{l^2\gamma * \gamma}{2} + \cdots\right)\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right)\\
&= -l\gamma\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) + \frac{l^2}{2}[\gamma(\bullet)]^2\\
\phi_{R,S}(\bullet) &= -l\gamma(\bullet).
\end{aligned}$$

Quickly, some notes before moving on to the next section.

While not necessary, we did some work with regularization, including an additional variable z. How can we see this additional structure and the anomalous dimension in this structure? Recall that $Y(f) = |f|f$. We saw that

$$_z\phi_S(f) = s^{-z|f|}\prod_{v \in V(f)} F(z|f_v|) = \ _z\phi_\mu(f) \cdot \underbrace{\left(\frac{S}{\mu}\right)^{-z|f|}}_{e^{-lz|f|} = \sum_{n\geq 0}\frac{-lz|f|)^n}{n!}}.$$

Then,

$$\begin{aligned}
\gamma &:= -d_l \mid_{l=0} \phi_{R,S} = d_l \mid_{l=0} \lim_{z\to 0} {}_z\phi_{R,S} \\
&= -d_l|_{l=0} \lim_{z\to 0} ({}_z\phi_\mu \circ S) * {}_z\phi_\mu(e^{-zlY}) \\
&= {}_z\phi_\mu \left(\sum_{n\geq 0} \frac{(-lzY)^n}{n!} f \right) = {}_z\phi_\mu(e^{-lzY} f) \\
&= \lim_{z\to 0} {}_z\phi_\mu(S * zY) \Rightarrow {}_z\phi_\mu(S*Y) \in \frac{1}{z}\mathbb{Q}[[z]].
\end{aligned}$$

Here, everything is formulated with a single application of the Feynman rules.

So what does this tell us? In general, the un-renormalized Feynman rules have high order poles in z. Here we see that if we apply this to $S * zY$ then we can take a limit and compute the anomalous dimension. This gives the last implication; that there are only single order poles.

<u>How to compute γ:</u>

Recall $\phi_R \circ B_+ = P \circ F(-d_l l)\phi_R$. Apply $-d_l|_{l=0}$, then

$$\begin{aligned}
\gamma \circ B_+ &= -d_l \sum_{n\geq -1} c_n (-d_l)^n \phi_R|_{l=0} \\
&= \sum_{n\geq -1} c_n \gamma^{*(n+1)} * \underbrace{\phi_R|_{l=0}}_{=e}
\end{aligned}$$

$$\begin{aligned}
\phi_R &= \exp_*(-l\gamma) = \exp_*(0) \\
d_l\phi_R &= -\gamma * \phi_R
\end{aligned}$$

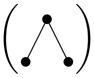
$$\gamma \circ B_+ = \sum_{n\geq -1} c_n \gamma^{*(n+1)}$$

Since we only need to consider trees, this gives a recursive procedure to define γ. That is, the B_+ operator on the left results in a tree with one additional node when a tree is input.

Example.

$$\gamma(\bullet) = \gamma \circ B_+(\mathbf{1}) = c_{-1} \overbrace{\gamma^{*0}}^{e}(\mathbf{1}) + c_0 \overbrace{\gamma(\mathbf{1})}^{0} + \cdots = c_{-1}$$

$$\phi_{R,S}(\bullet) = -c_{-1} l = \exp_*(-l\gamma)(\bullet) = -l\gamma(\bullet)$$

$\gamma\Bigg($  $\Bigg)$

$$= \gamma \circ B_+(\bullet\bullet) = c_0 \underbrace{\gamma(\bullet\bullet)}_{0} + c_1 \underbrace{(\gamma*\gamma)(\bullet\bullet)}_{\gamma(\bullet)\gamma(\bullet)} = c_1 c_{-1}^2$$

In all cases, the higher order terms vanish.

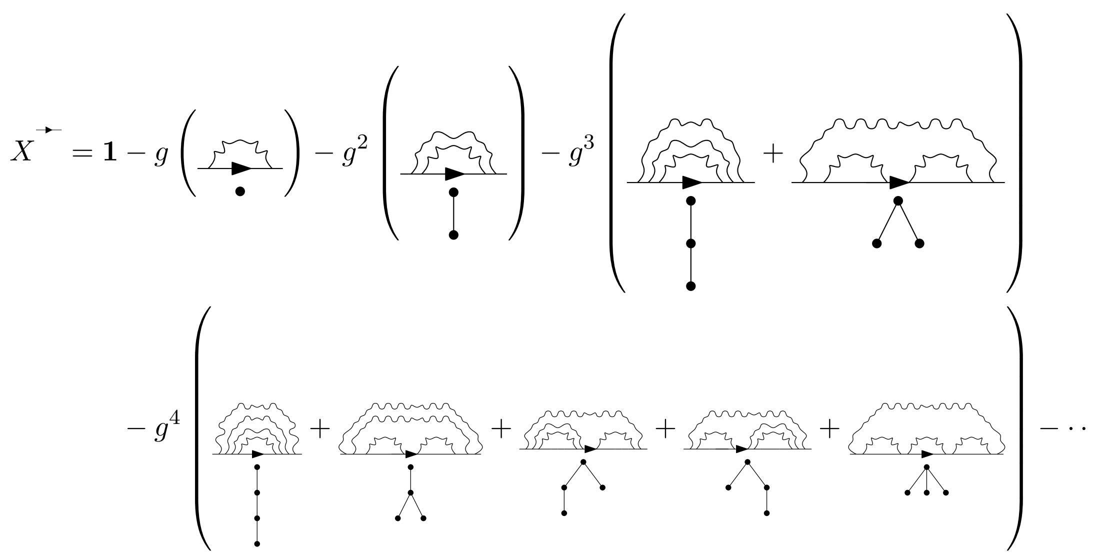
So, the general idea is; we have our Feynman rules, we compute the Mellin transform, with this recursion we can compute γ, and with this we can compute the renormalized Feynman rules by the convolution exponential.

1.1 Dyson-Schwinger equations

So far, we have only looked at the renormalization of Feynman rules and how they act on individual objects. In physics, though, we do not consider a single diagram, but the sum of infinitely many diagrams, and only those have a physical meaning. So we need to look at what these Feynman rules to the full infinite series of Feynman diagrams. But physics needs perturbation series.

This is also interesting for purely combinatorial reasons; even though it is physically motivated, it is a particular kind of generating series for combinatorial objects and in some cases we can get differential equations for these generating functions, for example.

Example. In QED;



The Feynman diagrams were chosen for their relation to this course; the trees drawn underneath capture the nesting of the photons.

Definition. A *perturbation series* is a series $X(g) = \sum_{n\geq 0} g^n x_n \in \mathcal{H}_R[[g]]$ (hence, dependent on the coupling constant). We call $\phi_{R,l}(X(g)) = G_l(g)$ the *Green's function* and $\gamma(X(g)) = \widetilde{\gamma}(g)$ the *physical anomalous dimension*.

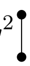
While these can be defined for any kind of series, but only some will be physically meaningful. An arbitrary perturbation series is not interesting.

Definition. Let $K \in \mathbb{Q}$ and $B_n : \mathcal{H}_R \to \mathcal{H}_R$ be a family of cocycles ($\Delta B_n = (\mathrm{id} \otimes B_n)\Delta + B_n \otimes \mathbf{1}$). We call $X(g) = \mathbf{1} + \sum_{n \geq 1} g_n B_n(X^{1+nK}(g))$ the *associated (combinatorial) Dyson Schwinger equation* (DSE).

We'll try some examples that turn out to be combinatorially meaningful.

Example ($K = 0$)**.**

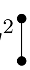
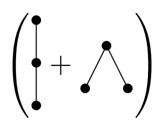
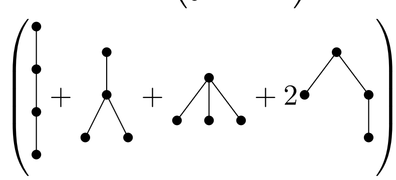
$$X(g) = \mathbf{1} + gB_+(X(g))$$

$$= \sum_{n \geq 0} g^n x_n = \mathbf{1} + g\bullet + g^2$$

$$+ \cdots = B_+^n(\mathbf{1})$$

This is the sum of ladders.

Example ($K = -2$)**.**

$$X(g) = \mathbf{1} - gB_+\left(\frac{1}{X(g)}\right)$$

$$= \mathbf{1} - g\bullet - g^2$$

$$- g^3$$

$$- g^4$$

$$- \cdots$$

The sum of the ordered trees.

It follows that these all have a unique solution, and can be solved by a simple induction. Further, the series we get from these construction are compatible with the coproduct.

Lemma 2. *Let $X = \mathbf{1} + \sum_n g^n B(X^{1+nK})$. Then,*

$$\Delta X = \sum_{n \geq 0} X^{1+nK} \otimes g^n x_n.$$

From the example, in the series generated by $K = 0$,

$$\begin{aligned}\Delta X &= \sum_{n\geq 0} g^n \Delta l_n \\ &= \sum_{n\geq 0}\sum_{k=0}^{n} g^k l_k \otimes l_{n-k} g^{n-k} \\ &= X \otimes X.\end{aligned}$$

In particular, we see that the physically meaningful perturbation series of all the ordered trees ($K = -2$) is captured by this theorem, and in this case we have a nice form of the coproduct of the perturbation series. This should remind you of the Feynman rules, which behave nicely with the coproduct, and now the perturbation series, the thing we apply the Feynman rules to, also behaves nicely with the coproduct. This leads us to the renormalization group.

1.2 Renormalization group

We now consider series of graphs. Let us look at $G_l(g) = \phi_{R,l}(X(g))$ for the solution $X(g)$ of a combinatorial DSE.

To try out a little example,

$$\begin{aligned}G_{l+l'}(G) &= (\phi_{R,l} * \phi_{R,l'}) \\ &= m(\phi_{R,l} \otimes \phi_{R,l'})\Delta X(g) \\ &= \sum_{n\geq 0} \phi_{R,l}(X^{1+nK})\phi_{R,l'}(g^n x_n)\end{aligned}$$

Inserting our formula for the coproduct, and considering the previous lemma, then;

$$\begin{aligned}\Delta X &= \sum_{n\geq 0} X^{1+nK} \otimes g^n x_n \\ &= \sum_{n\geq 0} [G_l(g)]^{1+nk} \otimes \phi_{R,l'}(g^n x_n) \\ &= G_l(g) \sum_{n\geq 0} \phi_{R,l'}(x_n)\{g[G_l(g)]^K\}^n \\ &= G_l(g) \cdot G_{l'}(gG_l^K(g)).\end{aligned}$$

This is one form of the renormalization group equation. Keep in mind that it is the Green's function that is the physically important thing; it tells you for which coupling constant and for which momentum we can achieve measurements in an experiment. If we change the momentum of the particle, it's the same as changing the coupling constant up to a multiplicative factor. As such, we may think of l' as a variable, affecting coupling.

We consider now the differential form. This brings back the anomalous dimension and Green's function.

Corollary 3. $d_l G_l(g) = -\widetilde{\gamma}(g[G_k(g)]^{K-1})G_l(g) = -\widetilde{\gamma}(1 + Kgd_g)G_l(g)$

Remark 4. Note that, $d_l \log G_l = -\widetilde{\gamma}(ge^{K \log G_l})$, and $G_0(g) = 1$ ($G_l(g)$ at $l = 0$) . That is, $\widetilde{\gamma}(g)$ determines G_l. (Recall $\phi_R = \exp_*(-l\gamma)$, $\phi_{R,0} = e$, $G = \phi_R \circ X(g)$.)

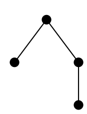
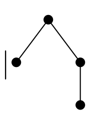
We bring this to an end with one instructive example.

When we use this for combinatorics, there are two things that must be considered; the perturbation series and the Feynman rules that you take.

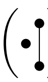
Example. Take tree-factorial Feynman rules;

$$\phi(f) = \frac{x^{|f|}}{f!}, \quad f! = \prod_{v \in V(f)} |f_v|.$$

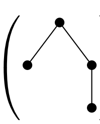
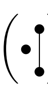
ie.

 $! = |\bullet| \cdot |\bullet| \cdot |$  $| \cdot |$  $| = 8.$

Hence,

$\phi\Big($  $\Big) = \phi(\bullet)\phi\Big($  $\Big) = x\frac{x^2}{2} = \frac{x^3}{2}.$

So, $\phi \circ B_+ = \int \circ \phi$.

$\phi\Big($  $\Big) = \int_0^x \phi_{x'}\Big($  $\Big)\, dx' = \int_0^x \frac{(x')^3}{2} dx' = \frac{x^4}{8}.$

So, the tree factorial fulfills the recursion, so we have a class of Feynman rules of the type we have been studying. It corresponds to the Mellin transform $F(z) = \frac{1}{z}$. If we compute the correlation function with these Feynman rules we are actually computing the sum of one over the tree factorials for all trees of the particular class we are considering.

For example, with $K = -2$, this was the sum over all ordered trees. Then, $z_\bullet(f) = \begin{cases} 1, f = \bullet \\ 0, \text{ else} \end{cases}$. We find that $\gamma = -d_l|_{l=0} \circ \phi = -z$, $\widetilde{\gamma} = \gamma \circ X(g) = -g$. Applying this to the perturbation series and taking the differential, $d_l G_l = -G_l \cdot g G_l^{-2} = -g\frac{1}{G_l(g)}$, and

$$d_l G_l^2 = -2g \Rightarrow G_l^2 - 1 = -2gl.$$

In other words, $G_l = \sqrt{1 - 2gl}$. To figure out what this computes, expand the equation in G, which you can do in terms of the Catalan numbers, you get the sum over all rooted trees with the number of nodes corresponding to the order in which you expand where each tree is weighted with one divided by it's tree factorial. So, this is a way to compute generating functions for such weighted sums of trees. This is known as the Hook weights in combinatorial literature.